Hurewicz Theorem: (simple version)

Thm:
- If X is $(n-1)$-connected ($n\geq 2$) then the reduced homology $\tilde{H}_i(X)=0$ for $i<n$, and $\pi_n(X)\simeq H_n(X)$
- If (X,A) is $(n-1)$-conn. ($n\geq 2$) and $A\neq\phi$ simply connected, then $H_i(X,A)=0$ for $i<n$ and $\pi_n(X,A)\simeq H_n(X,A)$.

(however no such nice relation between H_i and π_i for $i>n$ ...)

* **Ex:** $H_i(S^n)=0$ for $0<i<n$ and $H_n(S^n)\simeq\pi_n(S^n)\simeq\mathbb{Z}$.

* Can be used to compute $\pi_2(X)$ even if X not simply connected, by computing for universal cover: $\pi_2(X)\simeq\pi_2(\tilde{X})\simeq H_2(\tilde{X})$.

* **Rmk:**
  - absolute version follows from relative one by setting $A=pt$; conversely by excision
  - rel. version assumes A simply-connected, and X as well (since (X,A) 1-connected); $\exists$ more complicated version that doesn't require this. (cf. below)
  - recall for π_1 and H_1: X connected $\Rightarrow$ H_1 = abelianization of π_1.

Pf: by CW-approximation, can assume X is a CW-complex and (X,A) is a CW-pair.
Relative case then reduces to absolute case since by excision, for a CW-pair
$$\pi_i(X,A)\simeq\pi_i(X/A)\ \forall i\leq n \quad \text{(seen last time)}$$
$$H_i(X,A)\simeq\tilde{H}_i(X/A)\ \forall i.$$
Moreover, in absolute case, CW-approx. $\Rightarrow$ can assume X = point $\cup$ (cells of dim. $\geq n$).
This immediately implies $\tilde{H}_i(X)=0\ \forall i<n$.
Cells of dim $\geq n+2$ have no impact on π_n or H_n $\Rightarrow$ reduce to case where
$X=\left(\bigcup_\alpha S^n_\alpha\right)\cup\left(\bigcup_\beta e^{n+1}_\beta\right)$, attached by basepoint-preserving maps $\varphi_\beta: S^n\to X^n$
[CW-approximation gives this].

As seen last time, $\pi_n(X)=\bigoplus_\alpha\mathbb{Z}/\langle[\varphi_\beta]\rangle$ (i.e. $\pi_n(X)\simeq$ cokernel of ∂ map $\underset{\simeq\bigoplus_\beta\mathbb{Z}}{\pi_{n+1}(X,X^n)}\longrightarrow\underset{=\bigoplus_\alpha\mathbb{Z}}{\pi_n(X^n)}$)

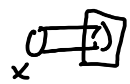
But cellular homology calc. gives same answer!
SINCE NO $(n-1)$-cells,
$H_n(X)$ = coker of cellular ∂ map $\underset{=C_{n+1}=\bigoplus_\beta\mathbb{Z}}{H_{n+1}(X^{n+1},X^n)}\xrightarrow{d}\underset{=C_n=\bigoplus_\alpha\mathbb{Z}}{H_n(X^n)}$, $d([e^{n+1}_\beta])=\sum c_{\alpha\beta}[S^n_\alpha]$,
$c_{\alpha\beta}$ = degree of φ_β onto S^n_α

$c_{\alpha\beta}$ = degree of $q_\alpha\circ\varphi_\beta$ (q_α = collapse all but S^n_α); since $\pi_n(S^n)\simeq\mathbb{Z}$ is given by degree, $c_{\alpha\beta}$ is <u>also</u> the coeff of $[\varphi_\beta]\in\pi_n(X^n)\simeq\bigoplus_\alpha\mathbb{Z}$ on summand α

Corollary: homology version of Whitehead's thm:

> A map $f: X \to Y$ b/w <u>simply conn.</u> CW-complexes is a homotopy eq^{ce} iff $f_*: H_n(X) \to H_n(Y)$ is an isomorphism $\forall n$.

<u>Pf:</u>
- after replacing Y by (homotopy equivalent) mapping cylinder $M_f = [0,1]\times X \cup Y/_\sim$ can assume $f =$ inclusion $X \hookrightarrow Y$



- X, Y simply connected $\Rightarrow$ (Y,X) is 1-connected too, and Hurewicz applies. So: the first nonzero $\pi_n(Y,X)$ is isomorphic to the first nonzero $H_n(Y,X)$.
- if f_* isoms. on H_n $\forall n$ then l.e.s. in homology $\Rightarrow$ $H_n(Y,X) = 0$ $\forall n$.
  Hence $\pi_n(Y,X) = 0$ $\forall n$
  l.e.s. in homotopy $\Rightarrow$ $\pi_n(X) \xrightarrow{f_*} \pi_n(Y)$ isom. $\forall n$
  Hence by Whitehead's thm, f is a homotopy eq^{ce}. $\blacktriangle$

- This statement is false when $\pi_1 \neq 0$, for at least 2 reasons:
  1) π_1 nonabelian and H_1 only sees its abelianization.
     (eg. Poincaré sphere $S^3/$(group of order 120 with trivial abelianization) minus point.
     has $H_* = H_*(\text{point})$ but nontrivial π_1.
  2) even when π_1 is abelian, the action of π_1 on π_n is missing from homology.

<u>Ex:</u> $X = (S^1 \vee S^n) \cup e^{n+1}$ where e^{n+1} is attached along map $S^n \to S^1 \vee S^n$ representing $2t - 1 \in \pi_n(S^1 \vee S^n) = \mathbb{Z}[t, t^{-1}]$

ie. on universal cover,

- univ. cover $\simeq \bigvee_\infty S^n \cup \left(\bigcup_\infty e^{n+1}\right)$
  $\Rightarrow \pi_n(X) \cong \mathbb{Z}[t,t^{-1}]/(2t-1)\,\mathbb{Z}[t,t^{-1}] \underset{\text{map } t \mapsto \frac{1}{2}}{\simeq} \mathbb{Z}[\tfrac{1}{2}] \subset \mathbb{Q}$

  (whereas $\pi_1(X) \simeq \pi_1(S^1) = \mathbb{Z}$, $\pi_i(X) = 0$ for $1 < i < n$)

- however, in homology, cellular boundary map
  $C_{n+1} = H_{n+1}(X, X^n) = \mathbb{Z}\cdot[e^{n+1}] \xrightarrow[d]{} C_n = H_n(X^n) = \mathbb{Z}\cdot[S^n]$ is an isom. since degree of attaching map onto S^n is $2 - 1 = 1$.
  so $H_n(X) = H_{n+1}(X) = 0$, in fact $H_i(X) = H_i(S^1)$ $\forall i$.

So incl. $S^1 \hookrightarrow X$ induces $\simeq$ on H_i $\forall i$, $\simeq$ on π_i for $i<n$ but <u>not</u> $i = n$. ($\Rightarrow$ not h.e).

<u>The Hurewicz map:</u> the isom. $\pi_n \simeq H_n$ in Hurewicz thm is induced by natural map which is always defined (but not always $\simeq$), the Hurewicz map

$$h : \pi_n(X, A, x_0) \longrightarrow H_n(X, A)$$

given $[f] \in$ homotopy class of $f : (D^n, \partial D^n) \to (X, A)$

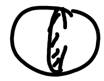
recall $H_n(D^n, \partial D^n) \simeq \mathbb{Z}$ with generator α (= the n-cell)

$$\rightsquigarrow \quad h([f]) = f_*[\alpha] \qquad f_* : H_n(D^n, \partial D^n) \to H_n(X, A).$$

Well-defined: If f, g homotopic then $f_* = g_*$ on $H_n \Rightarrow h([f]) = h([g])$ ✓.

<u>Prop:</u> $h : \pi_n(X, A, x_0) \to H_n(X, A)$ is a group homomorphism $(n > 1)$
so $\pi_n(X,A)$ group.

<u>Pf:</u> need to show $(f+g)_*$ induces $f_* + g_*$ on homology.

let $c : D^n \to D^n \vee D^n$ collapse  $\to$ point, then $f + g \simeq (f \vee g) \circ c$.

D^{n-1}

$$\begin{tikzcd}
H_n(D^n, \partial D^n) \arrow[r, "c_*"] \arrow[d, phantom, "\cong" rotate=90] & H_n(D^n \vee D^n, \partial D^n \vee \partial D^n) \arrow[r, "(f\vee g)_*"] \arrow[d, shift right, "q_{1*}\oplus q_{2*}"', "\simeq"] & H_n(X, A) \\
\mathbb{Z} & H_n(D^n, \partial D^n) \oplus H_n(D^n, \partial D^n) \arrow[u, shift right, "i_{1*}\oplus i_{2*}"', "\simeq"] &
\end{tikzcd}$$

$q_{1*}\oplus q_{2*}$ quotient map; $i_{1*}\oplus i_{2*}$ inclusion

$\simeq \mathbb{Z} \oplus \mathbb{Z}$

$1 \mapsto (1, 1)$

$(1, 0) \mapsto h([f])$

$(0, 1) \mapsto h([g])$

since $(f \vee g) \circ i_1 = f$
$(f \vee g) \circ i_2 = g$.

$\Rightarrow (f+g)_* :$ gen. $\mapsto h([f]) + h([g])$ ▲

- Similarly, <u>absolute case</u>: $h : \pi_n(X, x_0) \to H_n(X)$ homomorphism for $n \geq 1$.

$$[f] \longmapsto h([f]) = f_*(\alpha)$$
$$f : S^n \to X \qquad \alpha \in H_n(S^n) \simeq \mathbb{Z} \text{ generator.}$$

Hurewicz maps fit into comm. diagram of l.e.s. for pair (X, A) & are natural wrt maps of spaces.

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_{n+1}(X, A) \arrow[r, "\partial"] \arrow[d, "h"] & \pi_n(A) \arrow[r] \arrow[d, "h"] & \cdots \\
\cdots \arrow[r] & H_{n+1}(X, A) \arrow[r, "\partial"] & H_n(A) \arrow[r] & \cdots
\end{tikzcd}$$

- Back to failure of Hurewicz for non simply connected spaces: the issue is that for $f : (S^n, s_0) \to (X, x_0)$, $h([f]) = f_*(\text{gen. of } H_n(S^n))$ is insensitive to change of base point and non basept.-preserving homotopies.

So if $[\gamma] \in \pi_1(X, x_0)$, $\gamma \cdot f$ and f have same image:
$$h([\gamma]\cdot[f]) = h([f]).$$

Hence elements of the form $[\gamma][f] - [f]$ are always in $\ker(h)$.

Eg. for $S^1 \vee S^n$,
$$h: \pi_n(S^1 \vee S^n) \to H_n(S^1 \vee S^n)$$
$$\mathbb{Z}[t^{\pm 1}] \to \mathbb{Z}$$
$$t^k \mapsto 1$$

Similarly in relative case: $h: \pi_n(S^1 \vee S^n, S^1) \to H_n(S^1 \vee S^n, S^1)$ (same as by l.e.s.)
$$= \mathbb{Z}[t^{\pm 1}] \qquad = \mathbb{Z}$$
$$t^k \mapsto 1$$

so even though $(S^1 \vee S^n, S^1)$ is $(n-1)$-connected, $h: \pi_n \to H_n$ not iso!

However: this is the only issue preventing Hurewicz when $\pi_1(A) \neq 0$:

<u>Define</u> $\pi_n'(X, A, x_0) := \pi_n(X, A, x_0) / \langle [\gamma][f] - [f] \rangle$, $[f] \in \pi_n(X, A, x_0)$, $[\gamma] \in \pi_1(A)$
for $n \geq 2$

Then h descends to $h': \pi_n'(X, A, x_0) \to H_n(X, A)$

<u>Thm (Hurewicz).</u> (X, A) $(n-1)$-connected pair of path-connected spaces, $n \geq 2$, $A \neq \emptyset$, then $H_i(X, A) = 0$ for $i < n$ and $h': \pi_n'(X, A, x_0) \to H_n(X, A)$ isomorphism.

(we won't prove this).

<u>Next topic: fiber bundles</u>